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Quaternionic and octonionic spinors. A classification

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ABSTRACT: Quaternionic and octonionic realizations of Clifford algebras and spinors are classified and explicitly constructed in terms of recursive formulas. The most general free dynamics in arbitrary signature space-times for both quaternionic and octonionic spinors is presented. In the octonionic case we further provide a systematic list of results and tables expressing, e.g., the relations of the octonionic Clifford algebras with the G_2 cosets over the Lorentz algebras, the identities satisfied by the higher-rank antisymmetric octonionic tensors and so on. Applications of these results range from the classification of octonionic generalized supersymmetries, the construction of octonionic superstrings, as well as the investigations concerning the recently discovered octonionic M-superalgebra and its superconformal extension.

KEYWORDS: M-Theory, Space-Time Symmetries, Extended Supersymmetry, Differential and Algebraic Geometry.

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1. Introduction

The unification program aiming at a unified description of the known interactions as well as a consistent quantum formulation for gravity, nowadays mostly points towards higher-dimensional supersymmetric theories. At present the most promising, however still conjectural, candidate should live in eleven dimensions and goes under the name of M-theory [1]. The theoretical (and phenomenological) consistency requirements put on any possible candidate for unification necessarily lead to a systematic investigation of the properties of Clifford algebras and spinors in space-times of arbitrary dimension and signature. Exploring in full generality the existence of specific algebraic relations (such as the identities necessary to prove the k-symmetry invariance in the GS formulation of the superstring, see [2]), which are technically relevant in the model construction, is a necessary preliminary mathematical step before any attempt to model building.

From a mathematical point of view, Clifford algebras have been classified in the sixties (see [3]). Some twenty years later the relation between supersymmetry and division algebras was analyzed in [4]. A systematic and very convenient presentation in physicists' notation

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of the classification for Clifford algebras and spinors, based on the three associative division algebras of the real, complex and quaternionic numbers ($\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$), can be found in [5]. This relatively old subject has been revived recently in a series of work [6]. The aim in this case was the classification (once again based on the $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ division algebras) of the generalized supersymmetries admitting the presence of tensorial bosonic central charges and going therefore beyond the standard HLS scheme [7]. The real-valued M-algebra underlying the M-theory is the most celebrated and possibly the most physically relevant example in this class of generalized supersymmetries. In the last few months it was pointed out in [8] and [9] that the M-algebra admits a consistent octonionic restriction with surprising properties, which will be discussed in the following.

The first attempt of introducing octonions in physics goes back to the works of Jordan [10]. More recently, and in connection with the specific program of unification through supersymmetry, we can cite a series of works [11, 12] devoted to the octonionic description of the superstring. Already in [4] some mathematical results concerning the relations of the octonions with the Lorentz and Jordan algebras are mentioned, while a more developed investigation of this topic is presented in [13]. Moreover, in several different works (see e.g. [14, 15]) the octonionic characterization of the seven sphere S^7 (regarded as a compactification space for the eleven-dimensional maximal supergravity) and the analysis of its properties were investigated.

Octonions are non-associative and can not be represented through matrices with the standard matrix product. Octonionic realizations of Clifford algebras have peculiar properties, the most noticeable perhaps is the fact that they do not generate the corresponding Lorentz group, but only its coset over G_2, the group of automorphisms of the octonions [14].

This work is devoted to a systematic investigation of the properties of the quaternionic and the octonionic realizations of the Clifford algebras. More specifically, in the first part we classify such realizations, also furnishing recursive algorithms to explicitly construct them. Later, quaternionic and octonionic spinors are introduced. The notion of Weyl projection, whenever applicable, for these two classes of spinors is defined. The consistency conditions for the existence of a free dynamics for quaternionic and octonionic spinors are fully investigated and classified. We produce a whole set of tables expressing the allowed space-times admitting kinetic or pseudokinetic, as well as massive or pseudomassive, terms in the free-spinors lagrangian. These results can be considered as quaternionic and octonionic extensions of previous classification schemes available for real-valued spinors, see [16]. Since quaternionic Clifford algebras and spinors can always be represented through real-valued matrices and column vectors, the tables presented in the quaternionic case can be recovered from suitably constraining the real-valued case in order to admit a quaternionic structure. The situation however is entirely different, for the motivations that we already recalled, in the octonionic case.

To fully appreciate the further results presented in this paper concerning the octonionic case, some clarifications are needed. The octonionic M-superalgebra in [8] does not contain the Lorentz algebra. Indeed, it is an octonionic supertranslation algebra. However in [9] (based on the F-theory dual picture of the M-algebra, realized in terms of $(2,10)$ Majorana-Weyl octonionic spinors) the octonionic superconformal algebra has been

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explicitly constructed. The latter is the octonionic counterpart of $OSp(1,64)$, the generalized superconformal algebra of the M-theory. This superalgebra contains in particular an $\mathrm{SO}(2,10)$ bosonic subalgebra which, by Inonü-Wigner contraction, produces Poincaré in 11 dimensions. In the octonionic case as well an Inonü-Wigner contraction can be straightforwardly carried out. It is worth stressing the fact that in the octonionic case the Lorentz algebra is broken, but not arbitrarily, since we obtain in its place its G_2 coset. For a generic octonionic description of a (p,q) spacetime we obtain in fact $\mathrm{SO}(p,q)/G_2$. At least in special cases, the latter admits a geometrical interpretation. For instance, the euclidean seven-dimensional case is associated with the seven-sphere S^7, through its description in terms of unit octonions, see formula (4.5) and the following discussion.

It is remarkable that the M-algebra arises in two (and only two, due to the absence of the complex and of the quaternionic structures) versions, the standard *real* M-algebra with its 528 bosonic components and the octonionic algebra, with its restricted number of 52 bosonic components, potentially linked with exceptional structures (exceptional Lie and exceptional Jordan algebras), see e.g. the considerations in [17]–[21]. The octonionic M-algebra presents surprising new features, like the non-independence of the $M1$, $M2$, $M5$ brane-sectors, symbolically expressed through $M5 = M1 + M2$, see [8]. This relation is a special application of the higher-rank antisymmetric octonionic tensor identities, which have been classified for the first time in this paper and for which we present explicit tables (the very special case used in [8] was not explicitly written). Not only in physics, but even in the mathematical literature these identities have not been discussed (at least, no obvious reference can be found).

For what concerns the free dynamics of the octonionic spinors, it is worth remembering that our results are immediately applicable to field theories formulated on the seven sphere S^7 (or to higher-dimensional field theories admitting the seven sphere S^7 as a compactification space). Once more, this follows from the octonionic description of the seven-sphere S^7.

Let us further mention that the classification of the consistency conditions for the free octonionic dynamics should be regarded as a first preliminary step towards the investigation of octonionic supersymmetric dynamical systems associated to the generalized octonionic supersymmetries. It is worth stressing the fact that, for what concerns the latter, for the time being just examples of such superalgebras, the ones seemingly more attractive on physical grounds, have been analyzed so far. A classification scheme is still in progress.

The paper is organized as follows. In the next section we review [5] the classification of Clifford algebras and spinors in terms of the associative division algebras. In section 3 we present a systematic construction of the real irreducible representations of the Clifford algebras. This paves us the way to introduce in section 4 the explicit construction of the associative quaternionic and the non-associative octonionic realizations of the Clifford algebras. In section 5 we introduce the necessary conventions to introduce the dynamics for real, quaternionic and octonionic spinors. In section 6 the results of [16] concerning the classification of the most general free dynamics for real spinors in arbitrary signature space-times are reviewed. In section 7 and 8 these results are extended to, respectively, quaternionic and octonionic spinors. Section 9 is devoted to present a list of identities, due to the non-associativity of the octonions, involving higher-rank antisym-

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metric octonionic tensors. In the next section 10 some applications of these last results to octonionic generalized supersymmetries and M-theory are mentioned. Finally, in the conclusions 11, we point out possible future developments of the line of investigation here presented.

2. On Clifford algebras and division algebras

For later convenience we review in this section, following [5], the classification of the Clifford algebras associated to the **R**,**C**, **H** associative division algebras.

The most general irreducible *real* matrix representations of the Clifford algebra

$$\Gamma^\mu\Gamma^\nu + \Gamma^\mu\Gamma^\nu = 2\eta^{\mu\nu}\,, \tag{2.1}$$

with $\eta^{\mu\nu}$ being a diagonal matrix of (p,q) signature (i.e. p positive, $+1$, and q negative, -1, diagonal entries), can be classified according to the property of the most general S matrix commuting with all the Γ's ($[S,\Gamma^\mu]=0$ for all μ). If the most general S is a multiple of the identity, we get the normal (**R**) case. Otherwise, S can be the sum of two matrices, the second one multiple of the square root of -1 (this is the almost complex, **C** case) or the linear combination of 4 matrices closing the quaternionic algebra (this is the **H** case). According to [5] the *real* irreducible representations are of **R**, **C**, **H** type, according to the table 1, whose entries represent the values $p-q$ mod 8.

R	**C**	**H**
$0,2$		$4,6$
1	$3,7$	5

Table 1: Division-algebra type of the real irreps of Clifford algebras in different space-times: normal (R), almost complex (C), quaternionic (H).

The real irreducible representation is always unique unless $p-q \mod 8 = 1,5$. In these signatures two inequivalent real representations are present, the second one recovered by flipping the sign of all Γ's ($\Gamma^\mu \mapsto -\Gamma^\mu$).

Furthermore, in the given signatures $p-q \mod 8 = 0,4,6,7$, without loss of generality, the Γ^μ matrices can be chosen block-antidiagonal (generalized Weyl-type matrices), i.e. of the form

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}. \tag{2.2}$$

In these signatures it is therefore possible to introduce the Weyl-projected spinors, whose number of components is half of the size of the corresponding Γ-matrices.[1]

The division algebra characteristic for spinors (of **R**, **C**, **H** type) can be found in [6].

It is useful to illustrate our discussion presenting a table with the division algebra characteristic and number of real components for both Clifford algebras ($\boldsymbol{\Gamma}$) and fundamental spinors ($\boldsymbol{\Psi}$), at least in the specific case of the minkowskian spacetimes up to 11 dimensions. We obtain the table 2.

It should be noticed that, as far as Clifford algebras are concerned, the table 2 is not symmetric under the exchange $(p,q)\leftrightarrow(q,p)$ (the simplest example is the one-dimensional

[1]It is worth mentioning here that our notion of *generalized* Weyl spinors differs from the one usually adopted, since the latter is employed in connection with *complex* representations of Clifford algebras, while we are working here with *real* representations (realized by matrices whose entries are real) of Clifford algebras. This point will be extensively discussed in the next section.

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(p,q)	$\mathbf{\Gamma}$	$\mathbf{\Psi}$	(p,q)	$\mathbf{\Gamma}$	$\mathbf{\Psi}$
$(1,0)$	$\mathbf{R},1$	$\mathbf{R},1$	$(0,1)$	$\mathbf{C},2$	$\mathbf{R},1$
$(1,1)$	$\mathbf{R},2$	$\mathbf{R},1$	$(1,1)$	$\mathbf{R},2$	$\mathbf{R},1$
$(1,2)$	$\mathbf{C},4$	$\mathbf{R},2$	$(2,1)$	$\mathbf{R},2$	$\mathbf{R},2$
$(1,3)$	$\mathbf{H},8$	$\mathbf{C},4$	$(3,1)$	$\mathbf{R},4$	$\mathbf{C},4$
$(1,4)$	$\mathbf{H},8$	$\mathbf{H},8$	$(4,1)$	$\mathbf{C},8$	$\mathbf{H},8$
$(1,5)$	$\mathbf{H},16$	$\mathbf{H},8$	$(5,1)$	$\mathbf{H},16$	$\mathbf{H},8$
$(1,6)$	$\mathbf{C},16$	$\mathbf{H},16$	$(6,1)$	$\mathbf{H},16$	$\mathbf{H},16$
$(1,7)$	$\mathbf{R},16$	$\mathbf{C},16$	$(7,1)$	$\mathbf{H},32$	$\mathbf{C},16$
$(1,8)$	$\mathbf{R},16$	$\mathbf{R},16$	$(8,1)$	$\mathbf{C},32$	$\mathbf{R},16$
$(1,9)$	$\mathbf{R},32$	$\mathbf{R},16$	$(9,1)$	$\mathbf{R},32$	$\mathbf{R},16$
$(1,10)$	$\mathbf{C},64$	$\mathbf{R},32$	$(10,1)$	$\mathbf{R},32$	$\mathbf{R},32$

Table 2: Division-algebra type and size of Clifford algebras (Gamma) and fundamental spinors (Psi) in different Minkowski spacetimes.

Clifford algebra with negative eigenvalue, represented by a 2×2 real matrix). On the other hand, the properties of spinors are invariant (in some of the cases, for the signatures allowing it, the Weyl projection is required). As a consequence, the theories under consideration can be equivalently described either working with the (p,q) or with the (q,p) signatures.

$\mathbf{\Psi}$	$\sharp$
$\mathbf{R}$ (32)	528
$\mathbf{C}$ (16)	256
$\mathbf{H}$ (8)	120
$\mathbf{O}$ (4)	52

Table 3: Maximal number of bosonic components in a supertranslation algebra for division algebra-valued supersymmetry generators with 32 real components.

For what concerns the generalized supersymmetry algebras, it should be pointed out that the notion of spin algebra, generalizing the standard notion of spin covering and based on the division algebra structure of spinors alone, has been introduced in [6]. On the other hand, a different prescription for constructing generalized supersymmetries is also possible and has been advocated in [8]. It requires matching the division algebra structures of both spinors and Clifford algebras. According to the table 3, e.g., in the 5-dimensional case a quaternionic structure can be imposed on the supersymmetry since both spinors and Clifford algebras are quaternionic. On the other hand in, let's say, the minkowskian 7-dimensional case, at most a complex structure can be imposed, because this is the minimal structure shared both by spinors and Gamma matrices (see [8] for details). We will come back later on this issue. For the time being, let us just present another table concerning the constraint generated by division-algebra structures on generalized supersymmetries. For the sake of clarity we will discuss fundamental spinors admitting 32 real components (as in the maximal supergravity or the 11-dimensional M-theory). Let us suppose that they admit a real, complex, quaternionic[2] or even an octonionic (as discussed later) division algebra structure. Accordingly, the supersymmetry generators Q_a can be represented, respectively, as

[2] as this is the case, e.g., for the 64-component euclidean $D = 11$ spinors. Quaternionic 32-component spinors exist for instance in the $(3,7)$ signature.

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32-dimensional real column vectors, 16-dimensional complex, 8-dimensional quaternionic or 4-dimensional octonionic spinors. The generalized supersymmetry algebra

$$\{Q_a, {Q_b}^*\} = \mathcal{Z}_{ab}\,, \tag{2.3}$$

where ${Q_a}^*$ denotes the principal conjugation in the given division algebra, admits a hermitean r.h.s. ($\mathcal{Z}_{ab} = \mathcal{Z}^*_{ba}$), given by a hermitean matrix $\mathcal{Z}_{ab}$ of, respectively, 32×32 real, 16×16 complex, 8×8 quaternionic or 4×4 octonionic-valued entries. Due to the hermiticity condition, in the different cases, the maximal number $\sharp$ of independent components for $\mathcal{Z}_{ab}$ is given by table 3.

It should be noticed that 528 is the number of saturated independent bosonic components in the M-algebra, deriving from the real structure of 11-dimensional minkowskian spinors. As it will be apparent in the following, an octonionic structure can be imposed on minkowskian 11-dimensional spinors, leading to an alternative, octonionic version of the M-algebra with only 52 independent bosonic components.

3. Clifford algebras revisited. Classification and explicit constructions

For our purposes it is convenient to review the classification of the irreducible representations of Clifford algebras from another point of view, making explicit an algorithm allowing to single out, in arbitrary signature space-times, a representative in each irreducible class of representations of Clifford's gamma matrices. As recalled in the previous section, the class of irreducible representations is unique apart special signatures, where two inequivalent irreducible representations are linked by sign flipping ($\Gamma^\mu \leftrightarrow -\Gamma^\mu$). The explicit construction presented here is the right tool allowing us to introduce, in the next section, the quaternionic and octonionic realizations for Clifford algebras and spinors.

Our construction goes as follows. At first we prove that starting from a given D spacetime-dimensional representation of Clifford's Gamma matrices, we can recursively construct $D+2$ spacetime dimensional Clifford Gamma matrices with the help of two recursive algorithms. Indeed, it is a simple exercise to verify that if γ_i's denotes the d-dimensional Gamma matrices of a $D = p+q$ spacetime with (p,q) signature (namely, providing a representation for the $C(p,q)$ Clifford algebra) then $2d$-dimensional $D+2$ Gamma matrices (denoted as Γ_j) of a $D+2$ spacetime are produced according to either

$$\begin{aligned}\Gamma_j \equiv & \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) \mapsto & \ (p+1, q+1)\,, \end{aligned} \tag{3.1}$$

or

$$\begin{aligned}\Gamma_j \equiv & \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \\ (p,q) \mapsto & \ (q+2, p)\,. \end{aligned} \tag{3.2}$$

It is immediate to notice, e.g., that the two-dimensional real-valued Pauli matrices τ_A, τ_1,

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1	$*$	2	$*$	4	$*$	8	$*$	16	$*$	32	$*$	64	$*$	128	$*$	256	$*$
$\underline{(1,0)}$	$\Rightarrow$	(2,1)	$\Rightarrow$	(3,2)	$\Rightarrow$	(4,3)	$\Rightarrow$	(5,4)	$\Rightarrow$	(6,5)	$\Rightarrow$	(7,6)	$\Rightarrow$	(8,7)	$\Rightarrow$	(9,8)	$\Rightarrow$
						(1,4)	$\to$	(2,5)	$\to$	(3,6)	$\to$	(4,7)	$\to$	(5,8)	$\to$	(6,9)	$\to$
					$\nearrow$												
				$\underline{(0,3)}$													
					$\searrow$												
						(5,0)	$\to$	(6,1)	$\to$	(7,2)	$\to$	(8,3)	$\to$	(9,4)	$\to$	(10,5)	$\to$
								(1,8)	$\to$	(2,9)	$\to$	(3,10)	$\to$	(4,11)	$\to$	(5,12)	$\to$
							$\nearrow$										
						$\underline{(0,7)}$											
							$\searrow$										
								(9,0)	$\to$	(10,1)	$\to$	(11,2)	$\to$	(12,3)	$\to$	(13,4)	$\to$
														(1,12)	$\to$	(2,13)	$\to$
													$\nearrow$				
												$\underline{(0,11)}$					
													$\searrow$				
														(13,0)	$\to$	(14,1)	$\to$
																(1,16)	$\to$
															$\nearrow$		
														$\underline{(0,15)}$			
															$\searrow$		
																(17,0)	$\to$

Table 4: Table with the maximal Clifford algebras (up to $d = 256$).

τ_2 which realize the Clifford algebra $C(2,1)$ are obtained by applying either (3.1) or (3.2) to the number 1, i.e. the one-dimensional realization of $C(1,0)$. We have indeed

$$\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3}$$

All Clifford algebras are obtained by recursively applying the algorithms (3.1) and (3.2) to the Clifford algebra $C(1,0)$ ($\equiv 1$) and the Clifford algebras of the series $C(0,3+4m)$ (with m non-negative integer), which must be previously known. This is in accordance with the scheme illustrated in the table 4.

Concerning the table 4, some remarks are in order. The columns are labeled by the matrix size d of the maximal Clifford algebras. Their signature is denoted by the (p,q) pairs. Furthermore, the underlined Clifford algebras in the table can be named as "primitive maximal Clifford algebras". The remaining maximal Clifford algebras appearing in the table are the "maximal descendant Clifford algebras". They are obtained from the primitive maximal Clifford algebras by iteratively applying the two recursive algorithms (3.1) and (3.2). Moreover, any non-maximal Clifford algebra is obtained from a given maximal Clifford algebra by deleting a certain number of Gamma matrices (as an example, Clifford algebras in even-dimensional spacetimes are always non-maximal).

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It is immediately clear from the above construction that the maximal Clifford algebras are encountered if and only if the condition

$$p - q = 1, 5 \bmod 8 \tag{3.4}$$

is matched.

The notion of Clifford's algebra of generalized Weyl type, namely satisfying the (2.2) condition, has already been introduced. All maximal Clifford's algebras, both primitive and descendant, are *not* of generalized Weyl type. As already recalled, the notion of generalized Weyl spinors is based on the *real* representations of the Clifford algebras which, for purpose of classification, are more convenient to use w.r.t. the *complex* representations that are usually employed. For this reason generalized Weyl spinors exist also in odd-dimensional space-time, see formula (2.2), while standard Weyl spinors only exist in even-dimensional spacetimes. This can be understood by analyzing a single example. The *real* irrep $C(0,7)$, with all negative signs, is 8-dimensional, see table 4, while the *real* irrep $C(7,0)$ is 16-dimensional, but of generalized Weyl type (2.2). Accordingly, the euclidean 8-dimensional fundamental spinors can be understood either as the 8-dimensional "Non-Weyl" spinors of $C(0,7)$, or the 8-dimensional "Weyl-projected" $C(7,0)$ spinors. On the other hand, allowing for *complex* representions, the sign flipping $C(0,7) \mapsto C(7,0)$ can be realized by multiplying all Gamma matrices by the imaginary unit "i". No doubling of the matrix size of the Γ's is found and the notion of Weyl spinors cannot be applied. One faces a similar situation in the one-dimensional spacetime. Working with *complex* representations, we can realize $C(1,0)$ with 1 and $C(0,1)$ with i (both one-dimensional). On the other hand, the *real* irreducible representation of $C(0,1)$ is 2-dimensional. It is explicitly given by the block-antidiagonal matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Throughout the text Weyl (Non-Weyl) spinors are always referred to the (2.2) property with respect to *real* representations of Clifford algebras. The non-maximal Clifford algebras are of Weyl type if and only if they are produced from a maximal Clifford algebra by deleting at least one temporal Gamma matrix which, without loss of generality, can always be chosen the one with diagonal entries.

Let us discuss now explicitly how non-maximal Clifford algebras are produced from the corresponding maximal Clifford algebras. The construction goes as follows. We illustrate at first the example of the non-maximal Clifford algebras obtained from the 2-dimensional maximal Clifford irrep $C(2,1)$ furnished by the three matrices τ_1, τ_2, τ_A given in (3.3). If we restrict the Clifford algebra to τ_1, τ_a, i.e. if we delete τ_2 from the previous set, we get the 2-dimensional irrep $C(1,1)$. If we further delete τ_1 we are left with τ_A only, which provides the 2-dimensional irrep $C(0,1)$ discussed above. On the other hand, deleting τ_A from $C(2,1)$ leaves us with τ_1, τ_2, the 2-dimensional irrep $C(2,0)$.

To summarize, from the 2-dimensional irrep of the "maximal Clifford algebra" $C(2,1)$ we obtain the 2-dimensional irreps of the non-maximal Clifford algebras $C(1,1)$, $C(0,1)$ and $C(2,0)$ through a "Γ-matrices deleting procedure". Please notice that, through deleting, we cannot obtain from $C(1,2)$ the irrep $C(1,0)$, since the latter is one-dimensional.

In full generality, non-maximal Clifford algebras are produced from the corresponding maximal Clifford algebras according to the table 5, which specifies the number of time-like or space-like Gamma matrices that should be deleted, as well as the generalized Weyl (W)

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W			NW		
(0 mod 8)	$\subset$	(1 mod 8)	(2 mod 8)	$\subset$	(1 mod 8)
(p,q)	$\Leftarrow$	$(p+1,q)$	(p,q)	$\Leftarrow$	$(p,q+1)$
(4 mod 8)	$\subset$	(5 mod 8)	(3 mod 8)	$\subset$	(1 mod 8)
(p,q)	$\Leftarrow$	$(p+1,q)$	(p,q)	$\Leftarrow$	$(p,q+2)$
(6 mod 8)	$\subset$	(1 mod 8)			
(p,q)	$\Leftarrow$	$(p+3,q)$			
(7 mod 8)	$\subset$	(1 mod 8)			
(p,q)	$\Leftarrow$	$(p+2,q)$			

Table 5: Explicit construction of the real irreps for non-maximal Clifford algebras from the maximal ones.

character or not (NW) of the given non-maximal Clifford algebra. We get table 5. In the entries of table 5 x *mod* 8 specifies the *mod* 8 residue of $t-s$ for any given (t,s) spacetime. The non-maximal Clifford algebras are denoted by $p \equiv t$, $q \equiv s$, while the maximal Clifford algebras are denoted by (p', q'), with $p' \geq p$, $q' \geq q$. The differences $p'-p$, $q'-q$ denote how many Clifford gamma matrices (of time-like or respectively space-like type) have to be deleted from a given maximal Clifford algebra to produce the irrep of the corresponding non-maximal Clifford algebra. To be specific, e.g., the 6 *mod* 8 non-maximal Clifford algebra $C(6,0)$ is obtained from the maximal Clifford algebra $C(9,0)$, whose matrix size is 16 according to 4, by deleting three gamma matrices.

To complete our discussion what is left is specifying the construction of the primitive maximal Clifford algebras for both the $C(0,3+8n)$ (which can be named as "quaternionic series", due to its connection with this division algebra, as we will see in the next section), as well as the "octonionic" series $C(0,7+8n)$. The answer can be provided with the help of the three Pauli matrices (3.3). We construct at first the 4×4 matrices realizing the Clifford algebra $C(0,3)$ and the 8×8 matrices realizing the Clifford algebra $C(0,7)$. They are given, respectively, by

$$C(0,3) \equiv \begin{array}{l} \tau_A \otimes \tau_1\,, \\ \tau_A \otimes \tau_2\,, \\ \mathbf{1}_2 \otimes \tau_A\,. \end{array} \tag{3.5}$$

and

$$C(0,7) \equiv \begin{array}{l} \tau_A \otimes \tau_1 \otimes \mathbf{1}_2\,, \\ \tau_A \otimes \tau_2 \otimes \mathbf{1}_2\,, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_1\,, \\ \mathbf{1}_2 \otimes \tau_A \otimes \tau_2\,, \\ \tau_1 \otimes \mathbf{1}_2 \otimes \tau_A\,, \\ \tau_2 \otimes \mathbf{1}_2 \otimes \tau_A\,, \\ \tau_A \otimes \tau_A \otimes \tau_A\,. \end{array} \tag{3.6}$$

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The three matrices of $C(0,3)$ will be denoted as $\overline{\tau}_i, = 1,2,3$. The seven matrices of $C(0,7)$ will be denoted as $\tilde{\tau}_i$, $i = 1,2,\ldots,7$.

In order to construct the remaining Clifford algebras of the two series we need at first to apply the (3.1) algorithm to $C(0,7)$ and construct the 16×16 matrices realizing $C(1,8)$ (the matrix with positive signature is denoted as γ_9, ${\gamma_9}^2 = \mathbf{1}$, while the eight matrices with negative signatures are denoted as γ_j, $j = 1,2\ldots,8$, with ${\gamma_j}^2 = -\mathbf{1}$). We are now in the position to explicitly construct the whole series of primitive maximal Clifford algebras $C(0,3+8n)$, $C(0,7+8n)$ through the formulas

$$C(0,3+8n) \equiv \begin{array}{ll} \overline{\tau}_i \otimes \gamma_9 \otimes \cdots & \cdots \quad \cdots \otimes \gamma_9\,, \\ \mathbf{1}_4 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \cdots & \cdots \cdots \otimes \mathbf{1}_{16}\,, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \cdots & \cdots \cdots \otimes \mathbf{1}_{16}\,, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \cdots & \cdots \cdots \otimes \mathbf{1}_{16}\,, \\ \cdots & \cdots \qquad \cdots\,, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \cdots & \cdots \otimes \gamma_9 \otimes \gamma_j\,, \end{array} \tag{3.7}$$

and similarly

$$C(0,7+8n) \equiv \begin{array}{ll} \tilde{\tau}_i \otimes \gamma_9 \otimes \cdots & \cdots \quad \cdots \otimes \gamma_9\,, \\ \mathbf{1}_8 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \cdots & \cdots \quad \cdots \otimes \mathbf{1}_{16}\,, \\ \mathbf{1}_8 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \cdots & \cdots \quad \cdots \otimes \mathbf{1}_{16}\,, \\ \mathbf{1}_8 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \cdots & \cdots \quad \cdots \otimes \mathbf{1}_{16}\,, \\ \cdots & \cdots \qquad \cdots\,, \\ \mathbf{1}_8 \otimes \gamma_9 \otimes \cdots & \cdots \otimes \gamma_9 \otimes \gamma_j\,, \end{array} \tag{3.8}$$

Please notice that the tensor product of the 16-dimensional representation is taken n times. The total size of the (3.7) matrix representations is then 4×16^n, while the total size of (3.8) is 8×16^n.

With the help of the formulas presented in this section we are able to systematically construct a set of representatives of the real irreducible representations of Clifford algebras in arbitrary space-times and signatures.

4. Quaternionic and octonionic realizations of Clifford algebras

In this section we discuss the relations of Clifford algebras with the division algebras of the quaternions (and of the octonions), from a slightly different point of view w.r.t. the one expressed in section 2.

The relation can be understood as follows. At first we notice that the three matrices appearing in $C(0,3)$ can also be expressed in terms of the imaginary quaternions τ_i satisfying

$$\tau_i \cdot \tau_j = -\delta_{ij} + \epsilon_{ijk}\tau_k\,. \tag{4.1}$$

As a consequence, the whole set of maximal primitive Clifford algebras $C(0,3+8n)$, as well as their maximal descendants, can be represented with quaternionic-valued matrices. In its turn the spinors have to be interpreted now as quaternionic-valued column vectors.

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Similarly, there exists an alternative realization for the Clifford algebra $C(0,7)$, obtained by identifying its seven generators with the seven imaginary octonions (for an updated review on the octonions see e.g. [22]) satisfying the algebraic relation

$$\tau_i \cdot \tau_j = -\delta_{ij} + C_{ijk}\tau_k\,, \tag{4.2}$$

for $i,j,k = 1,\ldots,7$ and C_{ijk} the totally antisymmetric octonionic structure constants given by

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{4.3}$$

and vanishing otherwise. This octonionic realization of the seven-dimensional euclidean Clifford algebra will be denoted as $C_{\mathbf{O}}(0,7)$. Due to the non-associative character of the (4.2) octonionic product (the weaker condition of alternativity is satisfied, see [23]), the octonionic realization cannot be represented as an ordinary matrix product and is therefore a distinct and inequivalent realization of this euclidean Clifford algebra with respect to the one previously considered (3.6). Please notice that, by iteratively applying the two lifting algorithms to $C_{\mathbf{O}}(0,7)$, we obtain matrix realizations (with octonionic-valued entries) for the maximal Clifford algebras of the series $C(n,7+n)$ and $C(8+n,n-1)$, for positive integral values of n ($n=1,2,\ldots$). The dimensionality of the corresponding octonionic-valued matrices is $2^n \times 2^n$. For completeness we should point out that the construction (3.8) leading to the primitive maximal Clifford algebras $C(0,7+8n)$, can be carried on with the help of an octonionic-valued realization of the γ_9 matrix. As a consequence, realizations of $C(0,7+8n)$ and their descendants can be produced acting on column spinors, whose entries are tensor products of octonions. In any case, in the following, we will focus on the single octonionic realizations $C_{\mathbf{O}}(n,7+n)$ and $C_{\mathbf{O}}(9+n,n)$ (here $n=0,1,2,\ldots$) which are of relevance in the context of the M-theory.

One should be aware of the properties of the non-associative realizations of Clifford algebras. In the octonionic case the commutators $\Sigma_{\mu\nu} = [\Gamma_\mu, \Gamma_\nu]$ are no longer the generators of the Lorentz group. They correspond instead to the generators of the coset $\mathrm{SO}(p,q)/G_2$, being G_2 the 14-dimensional exceptional Lie algebra of automorphisms of the octonions. As an example, in the euclidean 7-dimensional case, these commutators give rise to $7 = 21-14$ generators, isomorphic to the imaginary octonions. Indeed

$$[\tau_i, \tau_j] = 2C_{ijk}\tau_k\,. \tag{4.4}$$

The alternativity property satisfied by the octonions implies that the seven-dimensional commutator algebra among imaginary octonions is not a Lie algebra, the Jacobi identity being replaced by a weaker condition that endorses (4.4) with the structure of a Malcev algebra (see [23]).

Such an algebra admits a nice geometrical interpretation [14, 15]. Indeed, the normed 1 unitary octonions $X = x_0 + x_i\tau_i$ (with x_0 and x_i, for $i=1,\ldots,7$, real and the summation over repeated indices understood), i.e. restricted by the condition

$$X^\dagger \cdot X = 1\,, \tag{4.5}$$

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describe the seven-sphere S^7. The latter is a parallelizable manifold with a quasi (due to the lack of associativity) group structure. Here $X^\dagger$ denotes the principal conjugation for the octonions, namely

$$X^\dagger = x_0 - x_i\tau_i\,. \tag{4.6}$$

On the seven sphere, infinitesimal homogeneous transformations which play the role of the Lorentz algebra can be introduced through

$$\delta X = a\cdot X\,, \tag{4.7}$$

with a an infinitesimal constant octonion. The requirement of preserving the unitary norm (4.5) implies the vanishing of the a_0 component, so that $a \equiv a_i\tau_i$. Therefore, the above commutator algebra (4.4), generated by the seven τ_i, can be interpreted as the algebra of "quasi" Lorentz transformations acting on the seven sphere S^7. At least in this specific example we discovered a nice geometrical setting underlining the use of the octonionic realization of the $C_{\mathbf{O}}(0,7)$ Clifford algebra. While the associative (3.6) representation of the seven dimensional Clifford algebra is required for describing the euclidean 7-dimensional flat space, the non-associative realization describes the geometry of S^7.

5. On real, quaternionic and octonionic spinors

In this section we introduce (following [4], where real and complex spinors were treated), the necessary ingredients and conventions to introduce the spinorial dynamics. Quaternionic and octonionic spinors are considered as well.

In [4] three matrices (only two of them independent) A, B, C, associated to the three conjugations (hermitean, complex and transposition) acting on Gamma matrices, were introduced. In the case of the restriction to real-valued Gamma matrices, only one matrix (conventionally denoted as A, see [16]) needs to be introduced. A plays the role of Γ^0 in the minkowskian case and serves to introduce barred spinors. In a (t,s) spacetime A is, up to a sign, the product of the time-like Gamma matrices and satisfies the relations

$$\begin{aligned} A\Gamma^\mu A^T &= \xi\Gamma^{\mu T}\,,\\ A^T &= \alpha A\,, \end{aligned} \tag{5.1}$$

with

$$\begin{aligned} \xi &= (-1)^{t-1}\,,\\ \alpha &= (-1)^{t(t-1)/2}\,, \end{aligned} \tag{5.2}$$

as it can be easily checked.

In both the quaternionic and octonionic case, two real-valued matrices, conventionally denoted as A and C, can be introduced. As before, A plays the role of Γ^0 and is used to define barred spinors ($\overline{\psi} = \psi^\dagger A$). A and C satisfy the set of relations

$$A\Gamma_\mu A^\dagger = \xi\Gamma_\mu^\dagger,$$

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$$
\begin{aligned}
C\Gamma_\mu C^\dagger &= \delta\Gamma_\mu^T\,,\\
C^T &= \rho C\,,\\
A^\dagger &= \alpha A\,,\\
A^T &= \sigma CAC^\dagger\,,
\end{aligned} \tag{5.3}
$$

where "†" denotes the combination of matrix transposition and principal conjugation in the division algebra (see (4.6)). The signs α, ξ, δ, ρ, σ will be specified below.

The matrix A is always given by the product of the temporal Γ's (regardless of the order), while up to two inequivalent C matrices can be found, given by the product (again, regardless of the order) of respectively all symmetric (C_S) or all antisymmetric (C_A) Gamma matrices (in special cases C_S, C_A collapse to the single matrix C).

For maximal Clifford algebras (in the sense specified in section 3) of a (t, s) space-time, the set of signs is given by

$$
\begin{aligned}
\alpha &= (-1)^{t(t-1)/2}\,,\\
\xi &= (-1)^{t-1}\,,\\
\delta &= (-1)^t\\
\rho &= (-1)^{t(t+1)/2}\,,\\
\sigma &= \sin\left(\frac{|t-s|\pi}{2}\right)(-1)^{\frac{t(t+1)}{2}+1}\,,
\end{aligned} \tag{5.4}
$$

as it can be checked with straightforward computations. Please notice that the matrix C is unique in this case.

The maximal quaternionic Clifford algebras are those satisfying the

$$t - s = 5 \bmod 8 \tag{5.5}$$

condition, while the maximal octonionic Clifford algebras are the subclass of

$$t - s = 1 \bmod 8 \tag{5.6}$$

maximal Clifford algebras, obtained after erasing the series corresponding to the first row in table 4 (i.e. $t = s + 1$).

Just like the real case, non-maximal Clifford algebras are obtained after erasing a certain number of Gamma matrices. The quaternionic equivalent of table 5 is given, in the quaternionic case, by table 6. while, in the octonionic case, we have the table 7. Please notice that the symbols appearing in the two tables 6 and 7 above have already been explained when introduced the table 5.

We should mention that, to be consistent, in, let's say, the octonionic realization of a non-maximal Clifford algebra, all the seven matrices proportional to the imaginary octonions must be present. Stated otherwise, the deleted matrices from the corresponding maximal Clifford algebra are all real-valued.

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W			NW		
(4 mod 8)	$\subset$	(5 mod 8)	(6 mod 8)	$\subset$	(5 mod 8)
(p,q)	$\Leftarrow$	$(p+1,q)$	(p,q)	$\Leftarrow$	$(p,q+1)$
(3 mod 8)	$\subset$	(5 mod 8)	(7 mod 8)	$\subset$	(5 mod 8)
(p,q)	$\Leftarrow$	$(p+2,q)$	(p,q)	$\Leftarrow$	$(p,q+2)$
(2 mod 8)	$\subset$	(5 mod 8)	(0 mod 8)	$\subset$	(5 mod 8)
(p,q)	$\Leftarrow$	$(p+3,q)$	(p,q)	$\Leftarrow$	$(p,q+3)$
(1 mod 8)	$\subset$	(5 mod 8)			
(p,q)	$\Leftarrow$	$(p+4,q)$			

Table 6: Explicit construction of the quaternionic irreps for non-maximal Clifford algebras from the maximal ones.

W			NW		
(0 mod 8)	$\subset$	(1 mod 8)	(2 mod 8)	$\subset$	(1 mod 8)
(p,q)	$\Leftarrow$	$(p+1,q)$	(p,q)	$\Leftarrow$	$(p,q+1)$
(7 mod 8)	$\subset$	(1 mod 8)	(3 mod 8)	$\subset$	(1 mod 8)
(p,q)	$\Leftarrow$	$(p+2,q)$	(p,q)	$\Leftarrow$	$(p,q+2)$
(6 mod 8)	$\subset$	(1 mod 8)	(4 mod 8)	$\subset$	(1 mod 8)
(p,q)	$\Leftarrow$	$(p+3,q)$	(p,q)	$\Leftarrow$	$(p,q+3)$
(5 mod 8)	$\subset$	(1 mod 8)			
(p,q)	$\Leftarrow$	$(p+4,q)$			

Table 7: Explicit construction of the octonionic realizations of non-maximal Clifford algebras from the maximal ones.

For completeness, let us right down the values of the signs entering (5.3) for the quaternionic and octonionic non-maximal Clifford algebra cases obtained by deleting a single Gamma matrix. In all four cases below two inequivalent C matrices are present and the suffix (S or$A$) specifies whether C_S or C_A is involved, while the signs α, ξ are given by (5.2). Furthermore, in all four cases below we get

$$\begin{aligned} \delta_S &= (-1)^t\,, \\ \delta_A &= (-1)^{t+1}\,. \end{aligned} \tag{5.7}$$

The remaining signs are given by

1. in the quaternionic 4 mod 8 (W) case,

$$\begin{aligned} \rho_S &= (-1)^{t(t+1)/2}\,, \\ \rho_A &= -(-1)^{t(t-1)/2}\,, \\ \sigma_S &= \sin\left((t-s)\frac{3\pi}{8}\right)(-1)^{\frac{t(t+1)}{2}}\,, \\ \sigma_A &= \sin\left((t-s)\frac{3\pi}{8}\right)(-1)^{\frac{t(t-1)}{2}}\,, \end{aligned}$$

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2. in the quaternionic 6 mod 8 (NW) case,

$$
\begin{aligned}
\rho_S &= (-1)^{t(t+1)/2}\,,\\
\rho_A &= (-1)^{t(t-1)/2}\,,\\
\sigma_S &= \sin\left(|t-s|\frac{3\pi}{4}\right)(-1)^{t(t+1)/2+1}\,,\\
\sigma_A &= \sin\left(|t-s|\frac{3\pi}{4}\right)(-1)^{t(t-1)/2+1}\,,
\end{aligned}
$$

3. in the octonionic 0 mod 8 (W) case,

$$
\begin{aligned}
\rho_S &= (-1)^{t(t+1)/2}\,,\\
\rho_A &= -(-1)^{t(t-1)/2}\,,\\
\sigma_S &= \sin\left((t-s)\frac{3\pi}{16}\right)(-1)^{t(t+1)/2}\,,\\
\sigma_A &= \sin\left((t-s)\frac{3\pi}{16}\right)(-1)^{t(t-1)/2}\,,
\end{aligned}
$$

4. and finally in the octonionic 2 mod 8 (NW) case,

$$
\begin{aligned}
\rho_S &= (-1)^{t(t+1)/2}\,,\\
\rho_A &= (-1)^{t(t-1)/2}\,,\\
\sigma_S &= \sin\left(|t-s|\frac{\pi}{4}\right)(-1)^{\frac{t(t+1)}{2}+1}\,,\\
\sigma_A &= \sin\left(|t-s|\frac{\pi}{4}\right)(-1)^{\frac{t(t-1)}{2}+1}\,.
\end{aligned}
$$

We remind that in the Weyl (W) case, the projectors $P_\pm$ can be introduced through

$$
\begin{aligned}
P_\pm &= \frac{1}{2}(\mathbf{1}_{2d} \pm \overline{\Gamma})\,,\\
\overline{\Gamma} &= \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}
\end{aligned}
\tag{5.8}
$$

and chiral (antichiral) spinors can be defined through

$$
\Psi_\pm = P_\pm \Psi\,. \tag{5.9}
$$

It is worth ending this section writing down, symbolically, the most general spinorial terms in a free lagrangian which can possibly (depending on the signature and dimensionality of the space-time) be encountered in our theories. It is sufficient to list such terms in the octonionic case. One trivially realizes how to employ the same symbols in the quaternionic and real cases as well.

Different massive terms can be found in the Weyl (W) case, i.e.[3]

$$
M_{//} = tr(\Psi_+^\dagger A \Psi_+)\,,
$$

[3] Here "tr" denotes the projection onto the octonionic identity, $tr(x_0 + x_i\tau_i) = x_0$. It coincides with the standard trace when we are restricting to the quaternionic subcase.

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$$
\begin{aligned}
M_{\perp} &= tr(\Psi_+^\dagger A\Psi_- + \Psi_-^\dagger A\Psi_+)\,,\\
M_{//T,S} &= tr(\Psi_+^\dagger A\Gamma_{T,S}\Psi_+)\,,\\
M_{\perp T,S} &= tr(\Psi_+^\dagger A\Gamma_{T,S}\Psi_- + \Psi_-^\dagger A\Gamma_{T,S}\Psi_+)\,,\\
M_{//J} &= tr(\Psi_+^\dagger AJ\Psi_+)\,,\\
M_{\perp J} &= tr(\Psi_+^\dagger AJ\Psi_- + \Psi_-^\dagger AJ\Psi_+)\,,\\
M_{//F} &= tr(\Psi_+^\dagger AF\Psi_+)\,,\\
M_{\perp F} &= tr(\Psi_+^\dagger AF\Psi_- + \Psi_-^\dagger AF\Psi_+)\,,
\end{aligned}
\tag{5.10}
$$

where Γ_T, Γ_S denote, in a non-maximal Clifford algebra case, the presence of an external (deleted from the set of maximal Gamma's) Gamma matrix of time, or respectively, space-like type. Similarly, J denotes the product of two such matrices, either time-like or space-like, while F denotes the product of three external matrices. No other massive symbols need to be introduced, as it will appear from the tables given in the following. In a NW-case, similar symbols can be introduced. However, since in this case no chiral (antichiral) spinors are defined, full spinors are present in the r.h.s. and the "//" and "$\perp$" suffices must be dropped.

In full analogy, the set of symbols in a Weyl (W) kinetic case are given by[4]

$$
\begin{aligned}
K_{//} &= \frac{1}{2}tr[(\Psi_+^\dagger A\Gamma^\mu)\partial_\mu\Psi_+] + \frac{1}{2}tr[\Psi_+^\dagger(A\Gamma^\mu\partial_\mu\Psi_+)]\,,\\
K_{\perp} &= \frac{1}{2}tr[(\Psi_+^\dagger A\Gamma^\mu)\partial_\mu\Psi_-] + \frac{1}{2}tr[\Psi_+^\dagger(A\Gamma^\mu\partial_\mu\Psi_-)] +\\
&\quad + \frac{1}{2}tr[(\Psi_-^\dagger A\Gamma^\mu)\partial_\mu\Psi_+] + \frac{1}{2}tr[\Psi_-^\dagger(A\Gamma^\mu\partial_\mu\Psi_+)]\,,\\
K_{//T,S} &= \frac{1}{2}tr[(\Psi_+^\dagger A\Gamma^\mu\Gamma_{T,S})\partial_\mu\Psi_+] + \frac{1}{2}tr[\Psi_+^\dagger(A\Gamma^\mu\Gamma_{T,S}\partial_\mu\Psi_+)]\,,\\
K_{\perp T,S} &= \frac{1}{2}tr[(\Psi_+^\dagger A\Gamma^\mu\Gamma_{T,S})\partial_\mu\Psi_-] + \frac{1}{2}tr[\Psi_+^\dagger(A\Gamma^\mu\Gamma_{T,S}\partial_\mu\Psi_-)] +\\
&\quad + \frac{1}{2}tr[(\Psi_-^\dagger A\Gamma^\mu\Gamma_{T,S})\partial_\mu\Psi_+] + \frac{1}{2}tr[\Psi_-^\dagger(A\Gamma^\mu\Gamma_{T,S}\partial_\mu\Psi_+)]\\
K_{//J} &= \frac{1}{2}tr[(\Psi_+^\dagger A\Gamma^\mu J)\partial_\mu\Psi_+] + \frac{1}{2}tr[\Psi_+^\dagger(A\Gamma^\mu J\partial_\mu\Psi_+)]\,,\\
K_{\perp J} &= \frac{1}{2}tr[(\Psi_+^\dagger A\Gamma^\mu J)\partial_\mu\Psi_-] + \frac{1}{2}tr[\Psi_+^\dagger(A\Gamma^\mu J\partial_\mu\Psi_-)] +\\
&\quad + \frac{1}{2}tr[(\Psi_-^\dagger A\Gamma^\mu J)\partial_\mu\Psi_+] + \frac{1}{2}tr[\Psi_-^\dagger(A\Gamma^\mu J\partial_\mu\Psi_+)]\,,\\
K_{//F} &= \frac{1}{2}tr[(\Psi_+^\dagger A\Gamma^\mu F)\partial_\mu\Psi_+] + \frac{1}{2}tr[\Psi_+^\dagger(A\Gamma^\mu F\partial_\mu\Psi_+)]\,,\\
K_{\perp F} &= \frac{1}{2}tr[(\Psi_+^\dagger A\Gamma^\mu F)\partial_\mu\Psi_-] + \frac{1}{2}tr[\Psi_+^\dagger(A\Gamma^\mu F\partial_\mu\Psi_-)] +\\
&\quad + \frac{1}{2}tr[(\Psi_-^\dagger A\Gamma^\mu F)\partial_\mu\Psi_+] + \frac{1}{2}tr[\Psi_-^\dagger(A\Gamma^\mu F\partial_\mu\Psi_+)]\,.
\end{aligned}
\tag{5.11}
$$

Please notice that, due to the non-associativity of the octonions, in the kinetic case we have to correctly specify the order in which the operations are taken. There is no such problem

[4]As before, analogous symbols are employed in the NW-case, by dropping the suffices "//" and "$\perp$".

	0	1	2	3
1		K	K,M	M
2	M_S	K	K, K_S, M	K_S, M, M_S
3	M_{S1}, M_{S2}, M_J, K_J	K,M_J	K, K_{S1},K_{S2},M	K_{S1},K_{S2},K_J,M, M_{S1},M_{S2}
5		K	K,M	M

Table 8: Type of kinetic and massive terms allowed in different spacetimes: the real W-case.

	0	1	2	3
0		$K_{//}$	$M_{//}$, $K_{\perp}$	$M_{\perp}$
4		$K_{//}$	$M_{//}$, $K_{\perp}$	$M_{\perp}$
6	$K_{//T1}$, $K_{//T2}$, $M_{//J}$, $K_{\perp J}$	$K_{//}$, $M_{//T1}$,$M_{//T2}$, $K_{\perp T1}$, $K_{\perp T2}$, $M_{\perp J}$	$M_{//}$, $K_{\perp}$, $M_{\perp T1}$, $M_{\perp T2}$	$K_{//J}$, $M_{\perp}$
7	$K_{//T}$	$K_{//}$, $M_{//T}$,$K_{\perp T}$	$M_{//}$, $K_{\perp}$, $M_{\perp T}$	$M_{\perp}$

Table 9: Type of kinetic and massive terms allowed in different spacetimes: the real NW-case.

in the massive case since the matrices Γ_T, Γ_S, J and F can always be chosen, without loss of generality, real. Therefore in (5.10) at most bilinear octonionic terms are present and the non-associativity of the octonions plays no role.

6. The real case revisited

In reference [16] the Majorana condition for complex spinors was analyzed and the list of different signature spacetimes allowing for kinetic, pseudokinetic, massive and/or pseudo-massive terms in the free-Majorana spinors lagrangians were presented. A slight generalization of these results can be produced in this section, based of the classification of real spinors that we have previously discussed (we notice,*en passant*, that the spinors we are dealing with here are, by construction, real, so that no Majorana condition, referring to a previous complex structure, needs to be imposed).

It is just a matter of lengthy, but straightforward computations, to produce a set of tables of the allowed, non-vanishing, free kinetic and massive terms in each given signature space-times. In the following tables, the columns are labeled by $t \bmod 4$, while the rows by $t-s \bmod 8$. The entries represent, simbolically, the allowed kinetic and/or massive terms (the precise meaning of the symbols is discussed at the end of the previous section). An empty space means that, neither a kinetic, nor a massive term is allowed for the corresponding space-time.

The first table 8 is produced for the real NW case.

The second table 9 is for the real W (Weyl) case.

7. Quaternionic spinors and their free dynamics. A classification

In this section we present the tables of allowed free kinetic and massive terms for quaternionic spinors. As in the previous section 6, the columns are labeled by $t \bmod 4$ and the rows by $t-s \bmod 8$, while the symbols used in the entries are explained at the end of section 5.

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	0	1	2	3
0	K_{J_j}, K_F, M_{S_j}, M_{J_j}	K, K_F, M_{J_j}	K, K_{S_j}, M, M_F	K_{S_j}, K_{J_j}, M, M_{S_j}, M_F
5		K	K, M	M
6	M_S	K	K, K_S, M	K_S, M, M_S
7	K_J, M_{S_i},M_J	K, M_J	K, K_{S_i}, M	K_{S_i} ,K_J, M, M_{S_i}

Table 10: Type of kinetic and massive terms allowed in different spacetimes: the quaternionic W-case.

	0	1	2	3
1	$K_{//T_j}, K_{\perp J_j}$, $M_{//J_j}$	$K_{//}, K_{\perp T_j}$, $M_{//F}, M_{//J_j}, M_{\perp J_j}$	$K_{//F}, K_{\perp}$, $M_{//}, M_{\perp F}, M_{\perp T_j}$	$K_{\perp F}, K_{//J_j}$, $M_{\perp}$
2	$K_{//T_i}, K_{\perp J}$, $M_{//J}$	$K_{//}, K_{\perp T_i}$, $M_{//T_i}, M_{\perp J}$	$K_{\perp}$, $M_{//}, M_{\perp T_i}$	$K_{//J}$, $M_{\perp}$
3	$K_{//T}$	$K_{//}, K_{\perp T}$, $M_{//T}$	$K_{\perp}$, $M_{//}, M_{\perp T}$	$M_{\perp}$
4		$K_{//}$	$K_{\perp}$, $M_{//}$	$M_{\perp}$

Table 11: Type of kinetic and massive terms allowed in different spacetimes: the quaternionic NW-case.

In the NW case we have 10.

In the W (Weyl) case we have 11. Please notice that in the two tables above the suffix "j" denotes the existence of three inequivalent choices for the corresponding matrices (e.g., the three distinct space-like matrices S_j), while the suffix "i" denotes the existence of two inequivalent choices. As previously discussed, this is in accordance with the signature of the given space-time. Therefore, the let's say, $t-s=0 \bmod 8$, $t=2 \bmod 4$ spacetime admits, besides the K kinetic term, three extra kinetic terms K_{S_j} associated to the three external space-like Gamma matrices S_j, $j=1,2,3$, existing in this case.

8. Octonionic spinors and their free dynamics. A classification

Here we present the tables of allowed free kinetic and massive terms for octonionic spinors. As in the two previous sections, the columns are labeled by $t \bmod 4$ and the rows by $t-s \bmod 8$, while the symbols used in the entries are explained at the end of section 5.

In the NW case we have 12. In the W (Weyl) case we have 13. As in the previous section, the suffices "i" and "j" takes two and respectively three distinct values. With these last tables we completed the classification of the allowed free lagrangians for spinors in different space-times.

9. Identities for higher rank antisymmetric octonionic tensors

As we have seen in the previous sections, octonionic spinors are associated with octonionic Clifford algebras. In their turn, these ones are given by the maximal octonionic Clifford

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	0	1	2	3
1		K	K, M	M
2	M_S	K	K, K_S, M	K_S, M, M_S
3	K_J,M_{S_i},M_J	K, M_J	K, K_{S_i}, M	K_{S_i}, K_J, M, M_{S_i}
4	K_{J_j}, K_F, M_{S_j}, M_{J_j}	K, K_F, M_{J_j}, M_F	K, K_{S_j}, M, M_F	K_{S_j}, K_{J_j}, M, M_{S_j}

Table 12: Type of kinetic and massive terms allowed in different spacetimes: the octonionic W-case.

	0	1	2	3
0		$K_{//}$	$K_\perp$, $M_{//}$	$M_\perp$
5	$K_{//T_j}, K_{\perp J_j}$, $M_{//J_j}, M_{\perp F}$	$K_{//}, K_{\perp T_j}$, $M_{//T_j}, M_{\perp J_j}$	$K_\perp, K_{//F}$, $M_{//}, M_{\perp T_j}$	$K_{//J_j}, K_{\perp F}$, $M_\perp, M_{//F}$
6	$K_{//T_i}, K_{\perp J}$, $M_{//J}$,	$K_{//}, K_{\perp T_i}$, $M_{//T_i}, M_{\perp J}$	$K_\perp$, $M_{//}, M_{\perp T_i}$	$K_{//J}$, $M_\perp$
7	$K_{//T}$	$K_{//}, K_{\perp T}$, $M_{//T}$	$K_\perp$, $M_{//}, M_{\perp T}$	$M_\perp$

Table 13: Type of kinetic and massive terms allowed in different spacetimes: the octonionic NW-case.

algebras, specified by the two sets of octonionic realizations for the signatures

$$C_{\mathbf{O}}(m, 7+8n+m)\,, \qquad C_{\mathbf{O}}(9+8n+m, m)\,, \tag{9.1}$$

with $n, m \geq 0$, together with the class of octonionic non-maximal Clifford algebras obtained from (9.1) by deleting a certain number of *real-valued* Gamma matrices. The reality restriction on these extra Gamma matrices (which cannot therefore contain imaginary octonions) puts a constraint on the space-time signatures admitting an octonionic description. For later convenience, it is useful to present the list of the whole class of octonionic spacetimes recovered from the maximal Clifford algebras of space-time dimension $D = t + s$ up to $D = 13$. The following table can be produced, with the columns labeled by D, the dimensionality of the spacetime. The maximal Clifford algebras are underlined. In each entry the octonionic dimensionality $\mathbf{d}_\Psi$ of the fundamental spinors is also reported. The signatures admitting, for each given spacetime dimension D, spinors of minimal octonionic dimensionality are denoted with a "$*$". Finally, the chain of reductions from a given maximal Clifford algebra is sketchily reported (please notice that the chain of reductions is not necessarily unique, for instance the $(10,1)$ signature can be produced by erasing a single Gamma matrix either from $(11,1)$ or from $(10,2)$). We get 14. We have already recalled in section 4 that for the (t,s) space-times allowing an octonionic description, due to octonionic non-associative identities, the algebra generated by the commutators between Gamma matrices is not the $\mathrm{SO}(t,s)$ Lorentz algebra, but its G_2 coset $\mathrm{SO}(t,s)/G_2$ [14]. We present here a generalization of this result consisting of a list of higher-rank antisymmetric octonionic tensorial identities. It is worth mentioning that these identities have striking applications which we shall discuss in the next section.

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7	8	9	10	11	12	13
$\underline{(0,7)}^*, \mathbf{1}$						
$(7,0)^*, \mathbf{1}$	$(8,0)^*, \mathbf{1}$	$\underline{(9,0)}^*, \mathbf{2}$				
	$(0,8)^*, \mathbf{1}$	$\underline{(1,8)}^*, \mathbf{2}$				
	$(7,1), \mathbf{2}$	$(8,1)^*, \mathbf{2}$	$(10,0), \mathbf{4}$ $(9,1)^*, \mathbf{2}$	$\underline{(10,1)}^*, \mathbf{4}$		
		$(0,9)^*, \mathbf{2}$ $(2,7), \mathbf{4}$	$(1,9)^*, \mathbf{2}$ $(2,8), \mathbf{4}$	$\underline{(2,9)}^*, \mathbf{4}$		
		$(7,2), \mathbf{4}$	$(8,2), \mathbf{4}$	$(11,0), \mathbf{8}$ $(9,2)^*, \mathbf{4}$	$(11,1), \mathbf{8}$ $(10,2)^*, \mathbf{4}$	$\underline{(11,2)}^*, \mathbf{8}$
			$(0,10), \mathbf{4}$ $(3,7), \mathbf{8}$	$(1,10)^*, \mathbf{4}$ $(3,8), \mathbf{8}$	$(2,10)^*, \mathbf{4}$ $(3,9), \mathbf{8}$	$\underline{(3,10)}^*, \mathbf{8}$

Table 14: The octonionic space-times induced by the maximal octonionic Clifford algebras (underlined).

$D=7$	$(0,7),\ (7,0)$
$D=8$	$(0,8),(8,0)$
$D=9$	$(0,9),(9,0),(1,8),(8,1)$
$D=10$	$(1,9),(9,1)$
$D=11$	$(1,10),(10,1),(2,9),(9,2)$
$D=12$	$(2,10),(10,2)$
$D=13$	$(3,10),(10,3),(2,11),(11,2)$

Table 15: The D-dimensional octonionic space-times admitting minimal octonionic spinors.

The identities under consideration are applicable to the space-time signatures which, for a given total dimension D, admit spinors of minimal octonionic dimensionality (up to $D=13$, these are the signatures denoted with a "*" in the table above). The generalization of this construction to dimensions $D>14$ is straightforward. Here however, both for simplicity and for physical relevance, we limit ourselves to discuss such identities up to $D=13$, namely for the following spacetimes 15. Please notice that in $D=8,10,12$ dimensions we are dealing with fundamental Weyl spinors.

It is worth mentioning that the above table has been complemented, for $D=13$, with the non-maximal octonionic Clifford algebras $(10,3)$, $(2,11)$, arising from the maximal ones at the level $D=15$.

In the above cases for $D=7,8$ the fundamental spinors are 1 (octonionic)-dimensional, 2-dimensional for $D=9,10$, four-dimensional for $D=11,12$ and finally 8-dimensional for $D=13$. The total number of octonionic hermitean $\mathcal{H}$ (antihermitean $\mathcal{A}$) components in a squared matrix of $\mathbf{d}_\Psi$ size is therefore given by 16.

The antisymmetric product of $k>2$ octonionic Γ-matrices must be consistently specified to take into account the non-associativity of the octonions. As we soon motivate, the correct prescription is taking the antisymmetrized product of k octonionic matrices Γ_i

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($i = 1, 2, \ldots, k$) to be given by

$$[\Gamma_1 \cdot \Gamma_2 \cdot \cdots \cdot \Gamma_k] \equiv \frac{1}{k!} \sum_{perm.} (-1)^{\epsilon_{i_1 \ldots i_k}} (\Gamma_{i_1} \cdot \Gamma_{i_2} \cdot \cdots \Gamma_{i_k}) \,, \tag{9.2}$$

where $(\Gamma_1 \cdot \Gamma_2 \ldots \cdot \Gamma_k)$ denotes the symmetric product

$$(\Gamma_1 \cdot \Gamma_2 \cdot \cdots \cdot \Gamma_k) \equiv \frac{1}{2}(.((\Gamma_1\Gamma_2)\Gamma_3 \cdots)\Gamma_k) + \frac{1}{2}(\Gamma_1(\Gamma_2(\cdots\Gamma_k)).) \,. \tag{9.3}$$

The usefulness of this prescription is due to the fact that the product

$$A[\Gamma_1 \cdot \Gamma_2 \cdot \ldots \cdot \Gamma_k] \,, \tag{9.4}$$

with A the matrix (product of the time-like Gamma matrices) introduced in section 5 has a definite (anti)-hermiticity property. The different (9.4) tensors, for different choices of the Gamma's, are all hermitean or antihermitean, depending only on the value of k (not of the Γ's themselves).

	$\mathcal{H}$	$\mathcal{A}$
$D = 7, 8$	1	7
$D = 9, 10$	10	22
$D = 11, 12$	52	76
$D = 13$	232	280

Table 16: Total number of hermitian and antihermitian octonionic bosonic components in D odd-dimensional spacetimes.

In the presence of the Weyl spinors, the above (9.4) tensors can be bracketed with the P_+ projection operator, see (5.8), to give

$$P_+ A[\Gamma_1 \cdot \Gamma_2 \cdot \ldots \cdot \Gamma_k] P_+ \,. \tag{9.5}$$

Once taken into account, from the algorithmic table 4 applied to the octonionic Clifford algebras, that out of the D Gamma matrices, 7 are proportional to the imaginary octonions, while the remaining $D - 7$ are purely real, it is a matter of straightforward computations to check the number of independent octonionic components both for (9.4) (in the NW spacetimes) and for (9.5) (in the Weyl spacetimes).

In odd-dimensions D we get the table 17, whose columns are labeled by the antisymmetric tensors rank k. The hermitean components are underlined.

Similarly, in the even-dimensional Weyl case, we have 18.

The above tables show the existence of identities relating higher-rank antisymmetric octonionic tensors. Let us discuss a specific example, which is perhaps the most physically relevant. In $D = 11$ dimensions the 52 independent components of an octonionic hermitean (4×4) matrix can be expressed either as a rank-5 antisymmetric tensors (simbolically denoted as "$M5$"), or as the combination of the 11 rank-1 ($M1$) and the 41 rank-2 ($M2$)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$D = 7$	$\underline{1}$	7	7	$\underline{1}$	$\underline{1}$	7	7	$\underline{1}$						
$D = 9$	$\underline{1}$	$\underline{9}$	22	22	$\underline{10}$	$\underline{10}$	22	22	$\underline{9}$	$\underline{1}$				
$D = 11$	1	$\underline{11}$	$\underline{41}$	75	76	$\underline{52}$	$\underline{52}$	76	75	$\underline{41}$	$\underline{11}$	1		
$D = 13$	1	13	$\underline{64}$	$\underline{168}$	267	279	$\underline{232}$	$\underline{232}$	279	267	$\underline{168}$	$\underline{64}$	13	1

Table 17: Number of hermitian (antihermitian) bosonic components for octonionic antisymmetric tensors in different space-times: the W-case.

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	0	1	2	3	4	5	6	7	8	9	10	11	12
$D=8$	$\underline{1}$	0	7	0	$\underline{1}+\underline{1}$	0	7	0	$\underline{1}$				
$D=10$	0	$\underline{10}$	0	22	0	$\underline{10}+\underline{10}$	0	22	0	$\underline{10}$	0		
$D=12$	1	0	$\underline{52}$	0	75	0	$\underline{52}+\underline{52}$	0	75	0	$\underline{52}$	0	1

Table 18: Number of hermitian (antihermitian) bosonic components for octonionic antisymmetric tensors in different space-times: the NW-case.

tensors. The relation between $M1+M2$ and $M5$ can be made explicit as follows. The 11 vectorial indices μ are split into 4 real indices, labeled by $a,b,c,\ldots$ and 7 octonionic indices labeled by $i,j,k,\ldots$. We get, on one side,

$$\begin{array}{cc} 4 & M1_a \\ 7 & M1_i \\ 6 & M2_{[ab]} \\ 4\times 7=28 & M2_{[ai]} \\ 7 & M2_{[ij]}\equiv M2_i \end{array}$$

while, on the other side,

$$\begin{array}{cc} 7 & M5_{[abcdi]}\equiv M5_i \\ 4\times 7=28 & M5_{[abcij]}\equiv M5_{[ai]} \\ 6 & M5_{[abijk]}\equiv M5_{[ab]} \\ 4 & M5_{[aijkl]}\equiv M5_a \\ 7 & M5_{[ijklm]}\equiv \widetilde{M}5_i \end{array}$$

which shows the equivalence of the two sectors, as far as the tensorial properties are concerned. Please notice that the correct total number of 52 independent components is recovered

$$52=2\times 7+28+6+4\,. \tag{9.6}$$

The octonionic equivalence of different antisymmetric tensors can be symbolically expressed, in odd space-time dimensions, through 19.

We end up this section by commenting that, for non-minimal spinors, the dependance on the rank k alone of the hermitean or antihermitean character of (9.4) and (9.5) is not mantained. To be explicit, in $D=8$ space-time dimension, the spinors associated to the $(1,7)$ signature are non-minimal (the number of their components is twice the number of components for fundamental $(8,0)$ and $(0,8)$ spinors). The $(1,7)$ Clifford algebra is obtained from the $(1,8)$ Clifford algebra after deleting a spacelike matrix Γ_S. For what concerns tensors, e.g. two sets of vectors are found, the ones obtained from Γ_μ (μ a vector index in $(1,7)$) are hermitean, while the ones obtained from the commutators $[\Gamma_\mu,\Gamma_s]$ are antihermitean.

$D=7$	$M0\equiv M3$
$D=9$	$M0+M1\equiv M4$
$D=11$	$M1+M2\equiv M5$
$D=13$	$M2+M3\equiv M6$
$D=15$	$M3+M4\equiv M0+M7$

Table 19: The equivalence of the different octonionic M-sectors in D odd-dimensional space-times.

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10. An application of the octonionic spinors. The octonionic M-algebra and the generalized supersymmetries

We shortly review here what is perhaps the most promising application of the octonionic spinors, i.e. their connection with the octonionic M-algebra (and superconformal M algebra, see [8, 9]), a specific example of a generalized octonionic supersymmetry. The identities for antisymmetric octonionic tensors play in this case a special role.

The generalized space-time supersymmetries are the ones going beyond the standard HLS scheme [7]. This implies that the bosonic sector of the Poincaré or conformal superalgebra no longer can be expressed as the tensor product structure $B_{\rm geom} \oplus B_{\rm int}$, where $B_{\rm geom}$ describes space-time Poincaré or conformal algebras and the remaining generators spanning $B_{\rm int}$ are Lorentz-scalars.

In the particular case of the minkowskian $D = 11$ dimensions, where the M-theory should be found, the following construction is allowed. The spinors are real and have 32 components.

As recalled in section 2, by taking the anticommutator of two such spinors the most general expected result consists of a 32×32 symmetric matrix with $32 + \frac{32\cdot 31}{2} = 528$ components. On the other hand, the standard supertranslation algebra underlining the maximal supergravity contains only the 11 bosonic Poincaré generators and by no means the r.h.s. saturates the total number of 528. The extra generators that should be expected in the right hand side are obtained by taking the totally antisymmetrized product of k Gamma matrices (the total number of such objects is given by the Newton binomial $\binom{D}{k}$). Imposing on the most general 32×32 matrix the further requirement of being symmetric, the total number of 528 is obtained by summing the $k = 1$, $k = 2$ and $k = 5$ sectors, so that $528 = 11 + 55 + 462$. The most general supersymmetry algebra in $D = 11$ can therefore be presented as

$$\{Q_a, Q_b\} = (A\Gamma_\mu)_{ab} P^\mu + (A\Gamma_{[\mu\nu]})_{ab} Z^{[\mu\nu]} + (A\Gamma_{[\mu_1\ldots\mu_5]})_{ab} Z^{[\mu_1\ldots\mu_5]} \qquad (10.1)$$

(where A is the real matrix introduced in section 5).

$Z^{[\mu\nu]}$ and $Z^{[\mu_1\ldots\mu_5]}$ are tensorial central charges, of rank 2 and 5 respectively. These two extra central terms on the right hand side correspond to extended objects [24, 25], the p-branes. The algebra (10.1) is called the M-algebra. It provides the generalization of the ordinary supersymmetry algebra recovered by setting $Z^{[\mu\nu]} \equiv Z^{[\mu_1\ldots\mu_5]} \equiv 0$.

On the other hand, in the same 11-dimensional minkowskian spacetime, we can impose, as we have seen, an octonionic structure, with fundamental spinors assumed to be 4-component octonionic valued. The generalized supersymmetry algebra (2.3) admits on the r.h.s. a hermitean 4×4 octonionic-valued matrix with up to 52 independent components. They can be expressed, from the previous section results, either as the $11 + 41$ bosonic generators entering

$$\mathcal{Z}_{ab} = P^\mu (A\Gamma_\mu)_{ab} + Z_{\mathbf{O}}^{\mu\nu} (A\Gamma_{\mu\nu})_{ab}\,, \qquad (10.2)$$

or as the 52 bosonic generators entering

$$\mathcal{Z}_{ab} = Z_{\mathbf{O}}^{[\mu_1\ldots\mu_5]} (A\Gamma_{\mu_1\ldots\mu_5})_{ab}\,. \qquad (10.3)$$

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Differently from the real case, the sectors specified by (10.2) and (10.3) are not independent [8], leading to an unexpected and far from trivial new structure in the octonionic M-algebra.

The octonionic results contained in the present paper should be regarded as the necessary background towards a classification of the octonionic generalized supersymmetries which is at present still missing.

It is worth mentioning that the equation (2.3) with $\mathcal{Z}_{ab}$ given by (10.2) corresponds to just an octonionic supertranslation algebra. However in [9] its octonionic superconformal algebra has been explicitly computed. It is the octonionic counterpart of the $OSp(1,64)$ generalized superconformal algebra of the M-theory. This superalgebra contains in particular an $\mathrm{SO}(2,11)$ bosonic subalgebra which, by dimensional reduction to $\mathrm{SO}(2,10)$ and further Inonü-Wigner contraction, produces Poincaré in 11 dimensions. Even if the Inonü-Wigner contraction has not been explicitly written down in the octonionic case, it is nevertheless a completely straightforward procedure to be carried out. As already recalled, in the octonionic case the Lorentz algebra is broken, but not arbitrarily. We obtain in its place the G_2 coset, $\mathrm{SO}(p,q)/G_2$. At least in special cases, the latter admits a geometrical interpretation (the seven-dimensional case is associated with the seven-sphere S^7, described by unit octonions, see the discussion contained in the section 4).

The higher-rank antisymmetric octonionic tensor identities have been classified for the first time in the present paper (a very special case was used, but not explicitly written, in [8]). Not only in the physical, even in the mathematical literature these identities have not been discussed (at least, no obvious reference can be found). We feel that a careful investigation is deserved to check whether octonionic spinors indeed play a role in association with M-theory, as well as the arising of exceptional structures, exceptional Lie and Jordan algebras, in this context. On the other hand, octonionic spinors have already found application in the context of string theory (see e.g. [11]). It is quite natural to find out, as done here, the consistency conditions for the octonionic spinors free dynamics. Due to the lack of associativity of the octonions, octonionic spinors have never been systematically investigated as in the present paper. However, it is worth remembering that our results can find immediate application in connection with field theories defined on the seven sphere S^7 (or, higher-dimensional field theories admitting the seven sphere S^7 as a compactification space).

Finally, for what concerns the $1D$ octonionic supersymmetries [26] applied to octonionic quantum mechanics, a classification is now available [27].

11. Conclusions

In this paper we made a systematic investigation of real, quaternionic and octonionic-valued Clifford algebras and spinors, presenting their classification, as well as constructive formulas to iteratively produce them. Tables have been given with the most general free dynamics satisfied by real, quaternionic and octonionic spinors in each space-time which supports them. All kinetic and massive terms have been listed.

For what concerns the octonionic case, by far the most intriguing due to the non-associativity, we further presented the systematic construction and derived a series of tables expressing the identities among higher rank antisymmetric octonionic tensors. We motivated this line of research with the attempt at classifying the generalized octonionic supersymmetries. A first example, hopefully physically relevant, consists of the octonionic M-algebra, with its striking properties induced by the mentioned identities.

For what concerns the quaternionic spinors, they also can appear in connection with generalized supersymmetries. One can read, e.g., from the results here presented, that in the euclidean $D = 11$ dimensions quaternionic-valued spinors are allowed. It looks promising to employ them to construct a quaternionic euclidean version of the M algebra (we are in fact planning to address this problem in the future).

Coming back to the octonionic spinors, we mention a further list of topics where they can possibly find application. At the end of section 3 we pointed out that the octonionic realization of the 7-dimensional euclidean Clifford algebra is related with the geometry of the seven sphere S^7. A question, which deserves being investigated, can be raised. Is the octonionic description of the M-theory somehow related to the particular compactification of the 11-dimensional M-theory down to $AdS_4 \times S^7$? This compactification corresponds to a natural solution for the 11-dimensional supergravity [28]. It would be interesting to check whether the tensorial identities found in the octonionic construction find a counterpart also in the $AdS_4 \times S^7$ special compactification geometry. On the other hand, one should try to understand the physical implications of the octonionic M-algebra also from a purely algebraic point of view. Being expressed by a 4-dimensional octonionic matrix, it is outside a Jordan algebra scheme [29]. This raises the question of its quantum-mechanical consistency, which implies understanding whether, and to which extent, is it possible to adapt the prescription of [29] to the present situation.

It is worth mentioning a different dynamical system [17], which can be called a "Jordan Matrix Chern-Simon theory", proposed as a unique model, being associated with the exceptional Jordan algebra $J_3(\mathbf{O})$ of 3×3 hermitean octonionic matrices. In this context it seems relevant addressing, for octonionic fields, the status of the spin-statistic theorem, in order to carefully revise it. Throughout this paper we have assumed the octonionic spinors being Grassmann, anticommuting fields. However, it cannot be a priori excluded that in the octonionic case this assumption could be relaxed.

We finally mention that the octonions can be held responsible for the existence of a bunch of exceptional structures in Mathematics. As an example the 5 exceptional Lie algebras can all be produced from the octonions via the Tits' construction [18]. A lot of activity is currently devoted to explore the relevance for Physics of these exceptional structures [19], see also [20]. The octonions seem the right tool to investigate such connections, see e.g. [21]. The recognized importance of this line of research strongly motivated us to systematically present here the fundamental properties of octonionic fields and spinors, as well as their non-trivial relations, as the ones discussed in section 9.

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